

From Simple Math — Chaos

One of the nice things about math is that if you follow the rules you always get the same results. Well... most of the time.

Let's try an experiment. Let's calculate $3.7x(1-x)$ over and over again starting with $x = 0.5$. I've done it here using Excel on my laptop and a pocket calculator. (I was lazy and didn't copy all the calculator's numbers until they started getting interesting.)

	Excel	Calculator			
	0.50000	0.50000	33	0.82635	
			34	0.53094	
1	0.92500		35	0.92146	
2	0.25669		36	0.26778	
3	0.70596		37	0.72547	
4	0.76805		38	0.73690	
5	0.65915		39	0.71735	
6	0.83129		40	0.75021	0.75023
7	0.51892		41	0.69336	
8	0.92368		42	0.78667	
9	0.26084		43	0.62095	
10	0.71338	0.71338	44	0.87088	
11	0.75654		45	0.41607	
12	0.68150		46	0.89893	
13	0.80312		47	0.33615	
14	0.58504		48	0.82567	
15	0.89824		49	0.53258	
16	0.33819		50	0.92107	0.92144
17	0.82812		51	0.26898	0.26783
18	0.52665		52	0.72753	0.72555
19	0.92237		53	0.73345	0.73677
20	0.26492	0.26492	54	0.72336	0.71758
21	0.72054		55	0.74041	0.74983
22	0.74505		56	0.71114	0.69406
23	0.70282		57	0.76005	0.78566
24	0.77279		58	0.67479	0.62307
25	0.64966		59	0.81196	0.86896
26	0.84213		60	0.56492	0.42131
27	0.49190		61	0.90941	0.90209
28	0.92476		62	0.30483	0.32681
29	0.25745		63	0.78406	0.81401
30	0.70733	0.70733	64	0.62644	0.56016
31	0.76596		65	0.86585	0.91161
32	0.66329				

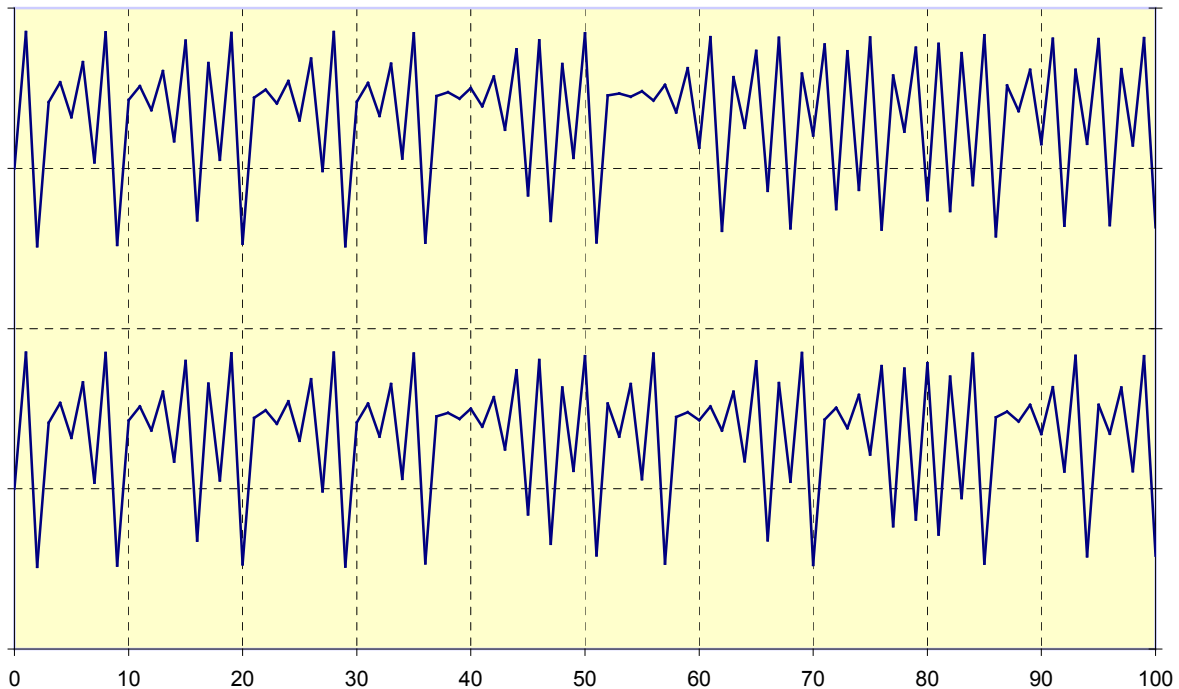
Starting around the 40th calculation, the number is different between Excel and the calculator. By the 50th calculation the last two digits are different. On the 60th, it's completely different.

So what's broken? My laptop or my calculator?

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Nothing's broken. Computers and calculators cannot represent numbers exactly. My calculator can only represent 10 digits so there's a tiny difference between the exact value of a number and the calculator's representation of the number. Excel does a little better, it keeps about 16 digits.

This function we're exploring is a **Chaotic Function**. This means that its behavior is extremely dependent on its initial value. This chart shows how a very small change in the starting value leads to completely different behavior after about 50 steps.



Mathematical Chaos:
The top plot starts with 0.50000; the lower plot starts with 0.50001

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There is another interesting feature of this function. Replacing the constant 3.7 with the variable r gives us $r x(1-x)$. How does the behavior of the function change as r varies $1 \leq r \leq 4$. Again, using Excel, we can experiment.

	2	2.5	2.9	3	3.1	3.5	3.7	3.83	3.9
	0.50000	0.50000	0.50000	0.50000	0.50000	0.50000	0.50000	0.50000	0.50000
1	0.50000	0.62500	0.72500	0.75000	0.77500	0.87500	0.92500	0.95750	0.97500
2	0.50000	0.58594	0.57819	0.56250	0.54056	0.38281	0.25669	0.15586	0.09506
3	0.50000	0.60654	0.70727	0.73828	0.76990	0.82693	0.70596	0.50390	0.33550
4	0.50000	0.59662	0.60041	0.57967	0.54918	0.50090	0.76805	0.95744	0.86946
5	0.50000	0.60166	0.69576	0.73096	0.76750	0.87500	0.65915	0.15606	0.44263
6	0.50000	0.59916	0.61387	0.58997	0.55317	0.38282	0.83129	0.50443	0.96217
7	0.50000	0.60042	0.68740	0.72571	0.76624	0.82694	0.51892	0.95742	0.14197
8	0.50000	0.59979	0.62316	0.59716	0.55527	0.50088	0.92368	0.15612	0.47508
9	0.50000	0.60010	0.68102	0.72168	0.76553	0.87500	0.26084	0.50459	0.97258
10	0.50000	0.59995	0.62998	0.60257	0.55643	0.38282	0.71338	0.95742	0.10401
11	0.50000	0.60003	0.67601	0.71844	0.76513	0.82694	0.75654	0.15614	0.36345
12	0.50000	0.59999	0.63516	0.60686	0.55709	0.50088	0.68150	0.50464	0.90228
13	0.50000	0.60001	0.67202	0.71574	0.76490	0.87500	0.80312	0.95742	0.34387
14	0.50000	0.60000	0.63919	0.61036	0.55747	0.38282	0.58504	0.15615	0.87993
15	0.50000	0.60000	0.66882	0.71346	0.76476	0.82694	0.89824	0.50466	0.41204
16	0.50000	0.60000	0.64235	0.61330	0.55770	0.50088	0.33819	0.95742	0.94483
17	0.50000	0.60000	0.66624	0.71149	0.76468	0.87500	0.82812	0.15615	0.20331
18	0.50000	0.60000	0.64486	0.61582	0.55783	0.38282	0.52665	0.50466	0.63170
19	0.50000	0.60000	0.66414	0.70976	0.76463	0.82694	0.92237	0.95742	0.90736
20	0.50000	0.60000	0.64686	0.61801	0.55790	0.50088	0.26492	0.15615	0.32783
1000	0.50000	0.60000	0.65517	0.65917	0.55801	0.50088	0.75092	0.95742	0.83563
1001	0.50000	0.60000	0.65517	0.67399	0.76457	0.87500	0.69204	0.15615	0.53568
1002	0.50000	0.60000	0.65517	0.65918	0.55801	0.38282	0.78855	0.50467	0.97004
1003	0.50000	0.60000	0.65517	0.67398	0.76457	0.82694	0.61694	0.95742	0.11336
1004	0.50000	0.60000	0.65517	0.65919	0.55801	0.50088	0.87440	0.15615	0.39198
1005	0.50000	0.60000	0.65517	0.67398	0.76457	0.87500	0.40635	0.50467	0.92950
1006	0.50000	0.60000	0.65517	0.65920	0.55801	0.38282	0.89255	0.95742	0.25558
1007	0.50000	0.60000	0.65517	0.67397	0.76457	0.82694	0.35486	0.15615	0.74201
1008	0.50000	0.60000	0.65517	0.65920	0.55801	0.50088	0.84705	0.50467	0.74658
1009	0.50000	0.60000	0.65517	0.67396	0.76457	0.87500	0.47935	0.95742	0.73788
1010	0.50000	0.60000	0.65517	0.65921	0.55801	0.38282	0.92342	0.15615	0.75431

For $r < 3$ the repeated function settles to a constant value.

For $r = 3$ it looks like it's settling to a fixed value but very slowly.

For $r = 3.1$ it gets stuck in a loop with 2 values.

For $r = 3.5$ it gets stuck in a loop with 4 values.

For $r = 3.7$ it behaves chaotically.

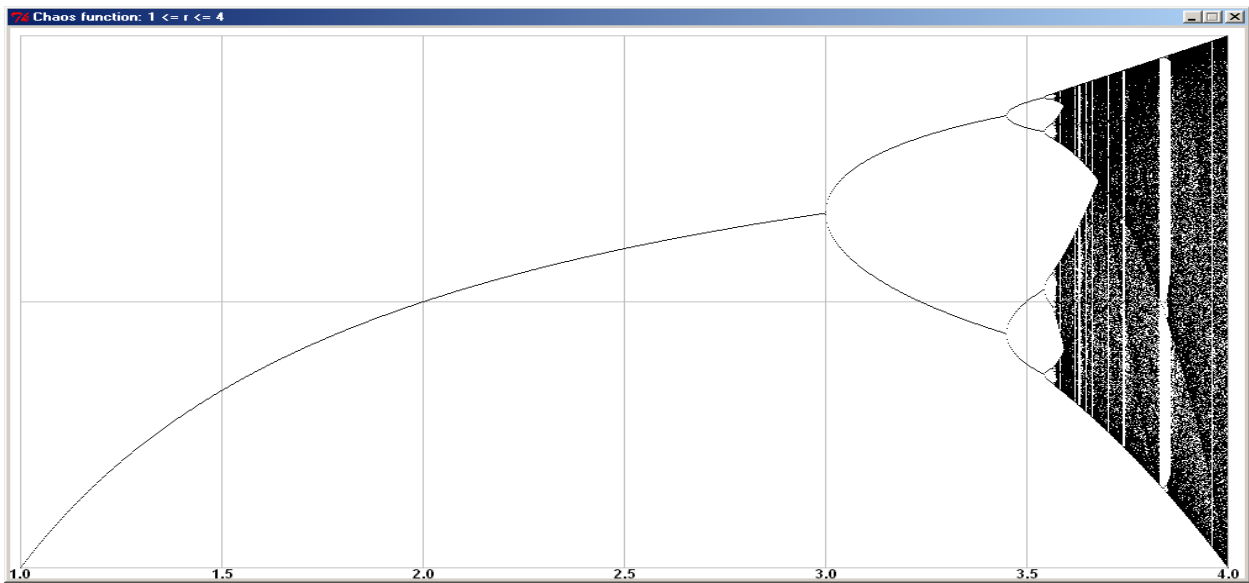
For $r = 3.83$ it gets stuck in a loop with 3 values.

For $r = 3.9$ it behaves chaotically.

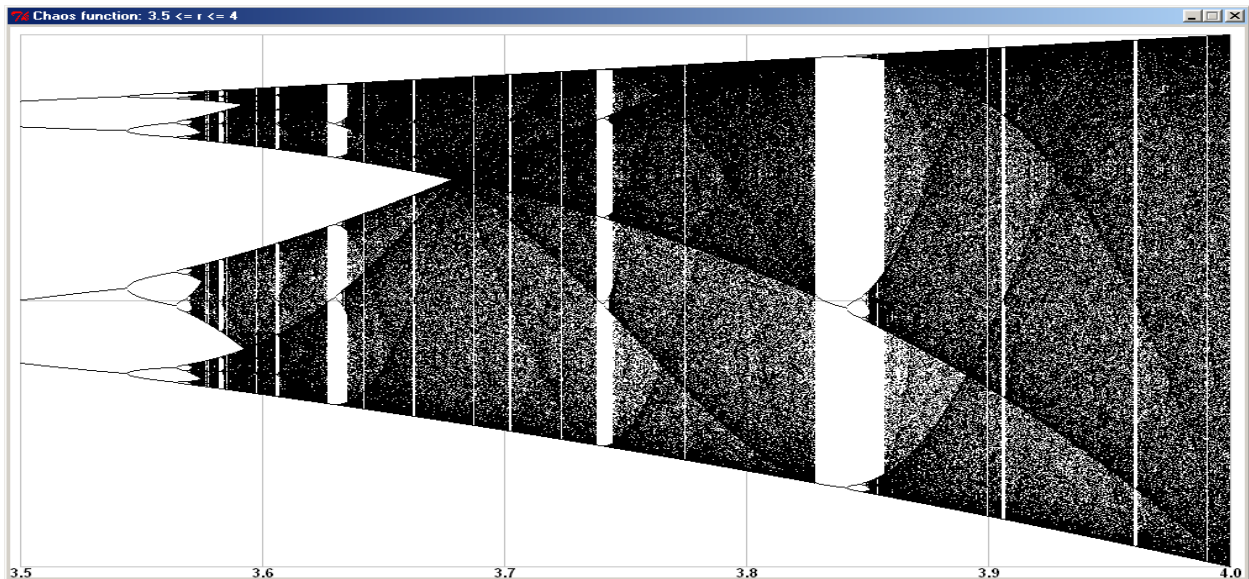
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This is a really strange function. Changing r can dramatically change its behavior.

The behavior the function settles into is called an **attractor**. If we make a graph with r on the horizontal axis and a lot of the sequential values of the function on the vertical axis, we can quickly see where changes in behavior occur. The following graphs skip the first 10,000 values and then plot the next 1,000. This way we see the attractor uncluttered by the function values as they approach the attractor. Mathematicians call this kind of graph a **bifurcation diagram**.



Attractor for $rx(1-x)$
 $1 \leq r \leq 4$



Attractor for $rx(1-x)$
 $3.5 \leq r \leq 4$

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It is easy to see on the detail graph that there are regions where the attractor is stable (either a fixed value or a multiple value cycle) and regions where it is chaotic.

Look at what happens when $r = 3$. The graph suddenly splits in two. Then at about $r = 3.55$ it splits in to four. It continues to double more and more often. If we looked close enough we'd see that when $r \approx 3.5699457$ the graph has split an infinite number of times and the function has become chaotic.

The black bands show where there is chaos and the white bands show where there is stability. It's interesting to see that there are regions of stability mixed in with the chaos.

If you look closely at the region around $r = 3.83$ you can see that each of the three branches follows the same doubling pattern that started at $r = 3.0$. So as r increases from 3.83 the attractor moves from 3-cyclic to 6-cyclic to 12, 24, 48...-cyclic and then breaks back to chaos.