

The Amazing Connection Between Exponents and Trigonometry

MIXING APPLES AND ORANGES

In the 17th century Brook Taylor determined how to generate a “power series” expansion for almost any function. Here are the Taylor series expansions for some functions we have been working with.

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$

$$\sin x = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

- 1) Write the first eight terms of the series expansion for e^{ix} by substituting (ix) for x in the first equation.
- 2) Expand the $(ix)^n$ terms and evaluate the powers of i . Note that there is a pattern to the values of i^n .
- 3) Group the real terms together and then group the imaginary terms together. The result should look like $e^{ix} = (\text{terms...}) + i(\text{terms...})$

If you've done everything correctly, the real terms should be the series expansion for $\cos x$ and the imaginary terms should be the series expansion for $\sin x$. Thus we can write

$$e^{ix} = \cos x + i \sin x$$

This is known as *Euler's formula* (pronounced **oil**-er), discovered in 1748 by Leonhard Euler. This amazing formula lets us write one of the more beautiful equations in all of math

$$e^{i\pi} + 1 = 0$$

which ties together five important constants.

- 4) What is the formula for e^{a+bi} ? Hint: use one of the power rules first and then use Euler's formula.

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AND YOU THOUGHT TRIGONOMETRY WAS COMPLEX

Using Euler's formula you can derive formulas for $\sin ix$ and $\cos ix$ in terms of e^x .

- 5) What is the formula for e^{-ix} ? Hint: remember that sine is an odd function and that cosine is an even function.

- 6) Use the formulas for e^{ix} and e^{-ix} to solve for $\sin x$ and $\cos x$. Hint: treat this as two equations in two unknowns where the unknowns are $(\sin x)$ and $(\cos x)$.

- 7) Replace x with (ix) in the $\sin x$ and $\cos x$ formulas to get formulas for $\sin ix$ and $\cos ix$.

To derive formulas for $\sin(a + bi)$ and $\cos(a + bi)$ you are going to need the sine and cosine angle addition formulas. In case you've forgotten them, they are easily derived using Euler's formula by substituting $(a+bi)$ for x :

$$\begin{aligned}\cos(a + b) + i \sin(a + b) &= e^{i(a+b)} \\ \cos(a + b) + i \sin(a + b) &= e^{ia} e^{ib} \\ \cos(a + b) + i \sin(a + b) &= (\cos a + i \sin a)(\cos b + i \sin b) \\ \cos(a + b) + i \sin(a + b) &= \cos a \cos b + i \cos a \sin b + i \sin a \cos b + i^2 \sin a \sin b \\ \cos(a + b) + i \sin(a + b) &= \cos a \cos b - \sin a \sin b + i(\cos a \sin a + \sin a \cos b)\end{aligned}$$

Since the real and imaginary parts of a complex number do not interact with each other, we can separate the last equation into two. We can also divide out the i in the expression for $\sin(a + bi)$.

$$\begin{aligned}\sin(a + b) &= \sin a \cos b + \cos a \sin b \\ \cos(a + b) &= \cos a \cos b - \sin a \sin b\end{aligned}$$

Mathematicians are lazy but have good memories. When they use the same equations over and over they give them names. Here are formulas for the *hyperbolic* sine, cosine and tangent:

$$\sinh x = \frac{e^x - e^{-x}}{2}, \quad \cosh x = \frac{e^x + e^{-x}}{2}, \quad \tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

These let you write $\sin ix$, $\cos ix$ and $\tan ix$ as

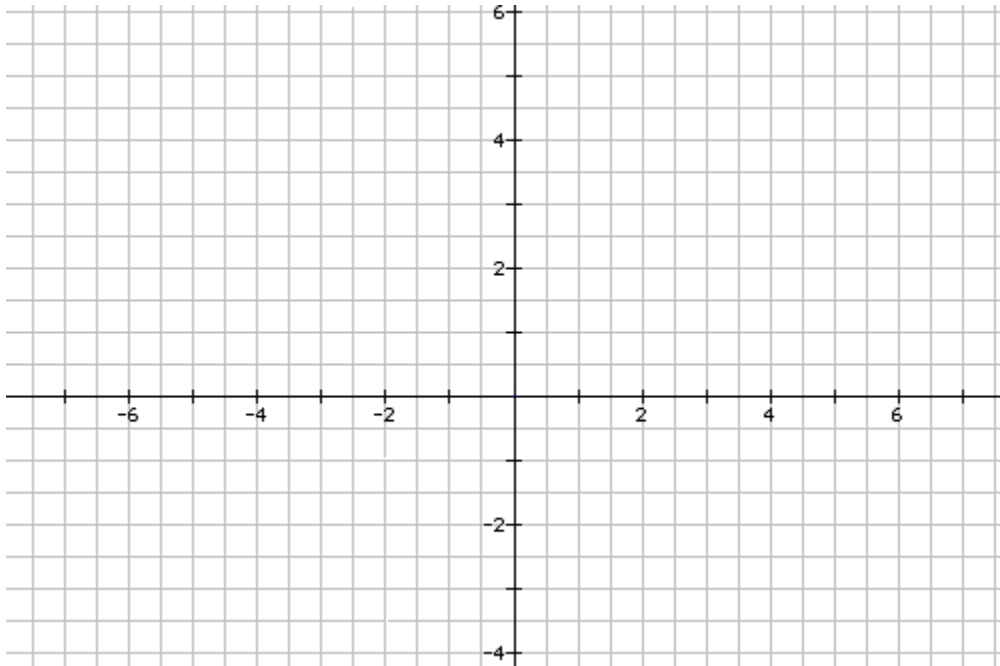
$$\sin ix = i \sinh x, \quad \cos ix = \cosh x, \quad \tan ix = i \tanh ix$$

- 8) What is $\cot ix$ in terms of $\coth x$?

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9) Derive formulas for $\sin(a + bi)$ and $\cos(a + bi)$

10) Graph the \sinh , \cosh , and \tanh functions.



RANGES? WE DON'T NEED NO STINKIN' RANGES!

As you can see, the real part of $\sin(a + bi)$ can be infinitely large. This suggests that $\sin^{-1} 2$ should exist as a complex number. You need to look for solutions to

$$2 = \sin(a + bi)$$

$$2 = \sin a \cosh b + i \cos a \sinh b$$

11) The imaginary term must be 0. For what values of a and b will $\cos a \sinh b = 0$?

12) Given the constraints on the values of a and b , what is to solution to $\sin a \cosh b = 2$?

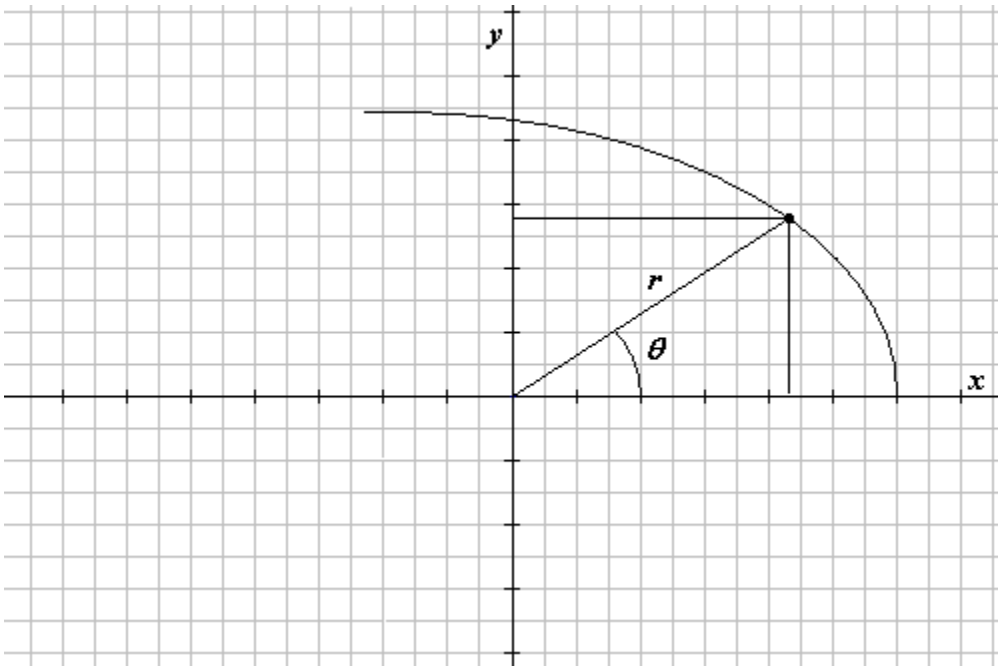
13) The a and b you just found are the real and imaginary components to the arcsine of 2!

$$\sin^{-1} 2 =$$

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ARCTIC EXPLORATION

Do you remember polar equations and polar graphs?



The conversion from polar form (r, θ) to rectangular form (x, y) is

$$x = r \cos \theta$$

$$y = r \sin \theta$$

If we replace the x axis with the real axis and the y axis with the imaginary axis, we can locate points on the complex plane with polar coordinates. (Here's another lazy mathematician shortcut: $z = a + bi$ then $\text{re}(z) = a$ and $\text{im}(z) = b$.)

$$\text{re}(z) = r \cos \theta$$

$$\text{im}(z) = r \sin \theta$$

$$z = r \cos \theta + i r \sin \theta$$

$$z = r(\cos \theta + i \sin \theta)$$

$$z = r e^{i\theta}$$

This gives us a compact way to write complex numbers in polar form that is easy to manipulate algebraically.

14) What is $r_1 e^{i\theta_1}$ times $r_2 e^{i\theta_2}$? Express your answer as a complex number in polar form.

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- 15) What is $(r e^{i\theta})^n$? Express your answer as a complex number in polar form.
- 16) Why might we want to work with complex numbers in polar form? Think about computing $(1 + i\sqrt{3})^{100}$ versus $(2 e^{i\pi/3})^{100}$. What is the value of $(2 e^{i\pi/3})^{100}$?
- 17) Can you think of geometrical rules for multiplication and exponentiation of complex numbers in polar form?
- 18) What is $\sqrt[n]{r e^{i\theta}}$? Remember that there are n solutions. Hint: in polar coordinates $\theta \equiv \theta + 2\pi \equiv \theta + 4\pi \equiv \dots$

THE POWER OF IMAGINATION

What is i^i ?

- 19) Think about the exponentiation rule for complex numbers. What is i in polar form? What is that value raised to the power of i ?
- 20) Remember that the polar form of a complex number is periodic in 2π . i^i is multi-valued. In fact, there are an infinite number of solutions. What are they?