Introduction to Complex Numbers

As you've been learning more and more math, you have been learning about more and more *types of numbers*:

Natural (counting) numbers	1, 2, 3, …
Whole numbers	**0**, 1, 2, 3, …
Integers	**…, -3, -2, -1,** 0, 1, 2, 3, …
Rational numbers (fractions)	*Add fractions to the integers.*
Irrational numbers (non-repeating decimal numbers)	*Numbers that are not fractions, for example* $\pi, \sqrt{2}$ *. These are non-repeating decimals: 3.14159…, 1.4142…*
Real numbers	*All the numbers on the number line.*

Whenever mathematicians made advances that needed a different type of number they met with resistance.

> Hippasus of Metapontum, one of the Pythagoreans, produced a proof of the irrationality of the square root of 2. According to one legend, Pythagoras believed in the absoluteness of numbers, and could not accept the existence of irrational numbers. He could not disprove their existence through logic, but his beliefs would not accept the existence of irrational numbers and so he sentenced Hippasus to death by drowning.
>
> from Wikipedia, "Square root of 2"

You've learned that negative numbers don't have square roots. There is no real number, *x*, that solves $x^2 = -2$.

In the 16th Century, there was a mathematician named Bombelli who wanted to solve a problem that he knew had a real number solution. The only formula that he knew of that solved his problem blew up in a middle step because it needed the square root of a negative number.

Mathematicians hate to say, "can't be done," so he took a bold step and assigned the bad square root to a variable and continued the formula using this variable and ignoring the fact that it had a really strange value. This happened twice in the formula and he kept going.

Eventually, the formula said to do some arithmetic with these weird values. When he carefully worked it out, the square roots of the negative values went away and he was left with the real number solution he needed!

Most mathematicians, of course, thought he was crazy!

He developed general rules for these weird numbers and eventually other mathematicians found that they were useful, too. Eventually these numbers were called **Complex Numbers**.

The square root of a negative number is called an **Imaginary Number**. The unit for imaginary numbers is the square root of -1. This unit is called ***i***. It works this way:

$$\sqrt{-4}$$
$$\sqrt{(4)(-1)}$$
$$\left(\sqrt{4}\right)\left(\sqrt{-1}\right)$$
$$2i$$

$2i$ is the imaginary number that is the square root of –4.

Real and imaginary numbers don't mix. Imagine them as red and blue beads; when you string them together you don't get purple beads. When you add a real number to an imaginary number you get a **Complex Number**. Complex numbers are written like an addition problem:

$2 + 3i$	a complex number,
$-3 + 0i$	a complex number that is a real number,
$0 - 1.2i$	a complex number that is an imaginary number.

How do you add complex numbers? The real and imaginary parts of the numbers do not mix. The real parts add together and the imaginary parts add together:

$$\begin{array}{r} a + bi \\ + \quad c + di \\ \hline (a + c) + (b + d)i \end{array}$$

Subtraction is similar:

$$\begin{array}{r} a + bi \\ - \quad c + di \\ \hline (a - c) + (b - d)i \end{array}$$

Multiplication is a little trickier. It's OK to multiply a real number and an imaginary number: $2 \times 3i = 6i$. Multiplying complex numbers follows the same pattern as a normal multiplication problem. The trick is to remember that $i^2 = -1$:

$$\begin{array}{rrr} & 1 & 2 \\ \times & 3 & 4 \\ \hline & 4 & 8 \\ 3 & 6 & \\ \hline 4 & 0 & 8 \end{array} \qquad \begin{array}{rr} a & + \ bi \\ \times \qquad c & + \ di \\ \hline adi & - \ bd \\ ac \quad + bci & \\ \hline (ac - bd) & + \ (ad + bc)i \end{array}$$

All complex numbers have a **conjugate**. The conjugate of $a + bi$ is $a - bi$. The conjugate is important because when you multiply a complex number by its conjugate you always get a real number: $(a + bi)(a - bi) = a^2 + b^2$.

Division is easiest to derive by first deriving the inverse since division is defined as:

$$\frac{x}{y} = x\frac{1}{y}$$

So let's tackle the complex inverse:

$$\begin{aligned}\frac{1}{a+bi} &= \frac{1}{a+bi}\cdot\frac{a-bi}{a-bi}\\ &= \frac{a-bi}{(a+bi)(a-bi)}\\ &= \frac{a-bi}{a^2+b^2}\end{aligned}$$

Now we can derive the formula for complex division:

$$\begin{aligned}\frac{a+bi}{c+di} &= (a+bi)\frac{1}{c+di}\\ &= (a+bi)\frac{c-di}{c^2+d^2}\\ &= \frac{(a+bi)(c-di)}{c^2+d^2}\\ &= \frac{ac+bd+bci-adi}{c^2+d^2}\\ &= \frac{ac+bd}{c^2+d^2}+\frac{bc-ad}{c^2+d^2}i\end{aligned}$$

Complex, isn't it!

The four basic operations for complex numbers:

$$\boxed{(a+bi)+(c+di)=(a+c)+(b+d)i}$$

$$\boxed{(a+bi)-(c+di)=(a-c)+(b-d)i}$$

$$\boxed{(a+bi)(c+di)=(ac-bd)+(ad+bc)i}$$

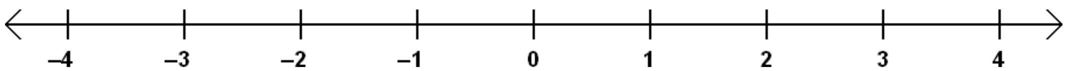
$$\boxed{\frac{a+bi}{c+di}=\left(\frac{ac+bd}{c^2+d^2}\right)+\left(\frac{bc-ad}{c^2+d^2}\right)i}$$

What's a complex number look like?

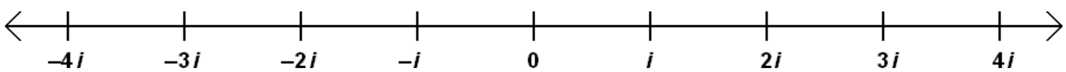
For that matter, what's a two look like? "2", "II" and "two" are all representations of two. "2 apples"? Nope—those are apples, not twos. Two is just a concept; there's no physical thing that's a two.

We have the concept of numbers so internalized that we don't ever think about it.

But we weren't born that way; we had to learn it. One of the tools used to teach numbers is the number line. You remember it from years ago in math class:

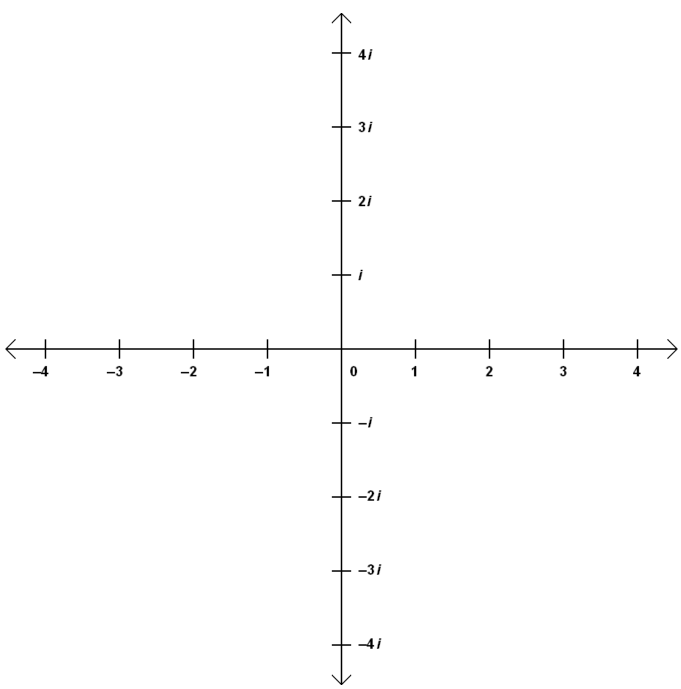


The imaginary number line looks just like it except that everything's multiplied by *i*:



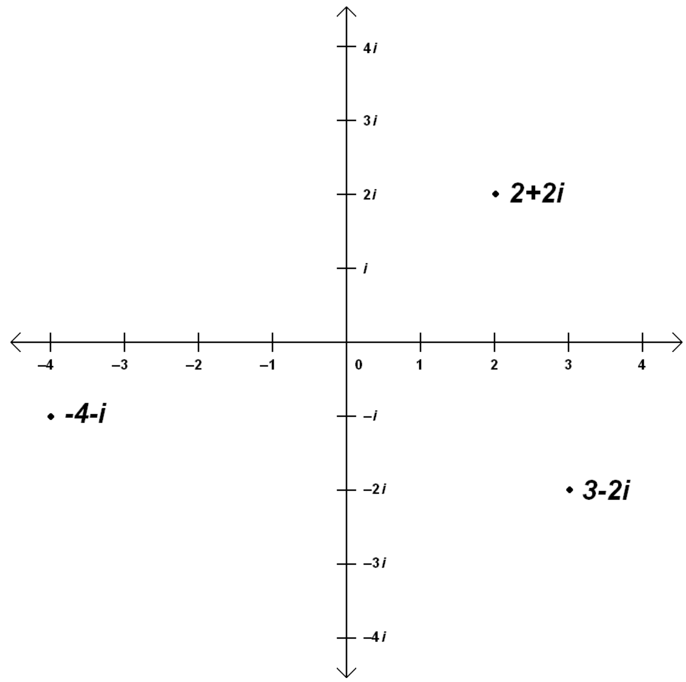
Notice that zero appears on both number lines. It's the same number, both real and imaginary. Is there a way to arrange the number lines so that the zeros are on top of each other but the real and imaginary numbers are not bothering each other? (Remember, they don't play well together; they want to be as far apart from each other as they can.)

Making the imaginary number line vertical and intersecting the real number line does this:



This is the **Complex Plane**. Any complex number can be placed uniquely on the complex plane. Coordinates on the complex plane work just like they do on normal graphs; that's the beauty of this arrangement.

Here are some numbers plotted on the complex plane:



Addition of two complex numbers has an interesting geometrical interpretation. Draw a parallelogram with zero at one corner and the two numbers at corners adjacent to zero. The fourth corner will be the sum. Here is an example showing $(2 + 2i) + (-4 - i) = -2 + i$.