The Sun Hemmi System of Trigonometric and Hyperbolic Scales

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Introduction

Most slide rules make use of S, T, and ST scales for trigonometric functions. When hyperbolic functions such as sinh and tanh are available, these are usually handled with a separate set of scales, such as the Sh1, Sh2, and Th scales. In this article we discuss a system of scales for trigonometric and hyperbolic functions which was used by Sun Hemmi in several of its slide rules. The Hemmi system makes it possible to solve for one side of a right triangle given the other two sides; evaluate sines, cosines, and tangents; and evaluate the hyperbolic functions sinh, cosh, and tanh using only seven scales.

Hemmi slide rules that used this system included the models 152, 153, 154, and 266. A version of the Hemmi 152 was sold as the Hughes Owen 1766. The Hemmi models 152, 153 (see Figures 2a and 2b), and 154 were sold in the United States by Post under the model numbers 1459, 1461, and 1460. These rules may also have been sold in the United States under the U.S. Blue and American Blueprint brand names. The Relay/Ricoh model 157 and 158 slide rules also used the Hemmi system for solving right triangles and for trigonometric functions, but not for hyperbolic functions. This Hemmi system was also used in the Lafayette F-686, 99-7102, and 99-71029 Vectorlog rules, which are very similar to the Relay 157. The Lafayette Vectorlog rules were apparently made by Relay/Ricoh. The V1 and V2 scales on the Pickett 110-ES circular slide rule have the same function as the P and Q scales in the Hemmi system.

According to a Hemmi catalog, the P and Q scales were invented by Drs. Sadatoshi Betsumiya and Jisuke Miyazaki in 1929 [1]. The Gθ scales for hyperbolic functions were invented by Hisashi Okura. A US patent was issued in 1937 for the Gθ scale [2].

Table 1 summarizes the scales on these rules. We will concentrate on the Hemmi 153 because it has the most complete set of scales within this system. However, everything in this article also applies to the other slide rules with slight changes. The special scales for trigonometric and hyperbolic functions on the Hemmi 153 are θ, Rθ, P, Q, Q', T, and Gθ. We begin by describing the P, Q, and Q' scales and their use in solving for one side of a right triangle given the other two sides. This calculation is particularly important in electrical engineering, where many calculations involve finding the length of a vector. We then show how the P and Q scales can be used along with the θ, Rθ, and T scales to compute trigonometric functions of angles in degrees and radians. Finally, we show how the Gθ scale can be used with the other scales to compute hyperbolic functions.

The Pythagorean Scales P, Q, and Q′

Consider a right triangle with sides of length a and b and hypotenuse of length c. From the Pythagorean theorem, we know that

$$a^2 + b^2 = c^2$$

See Figure 1. The P, Q, and Q' scales can be used to solve for any side of the triangle given the lengths of the other two sides.

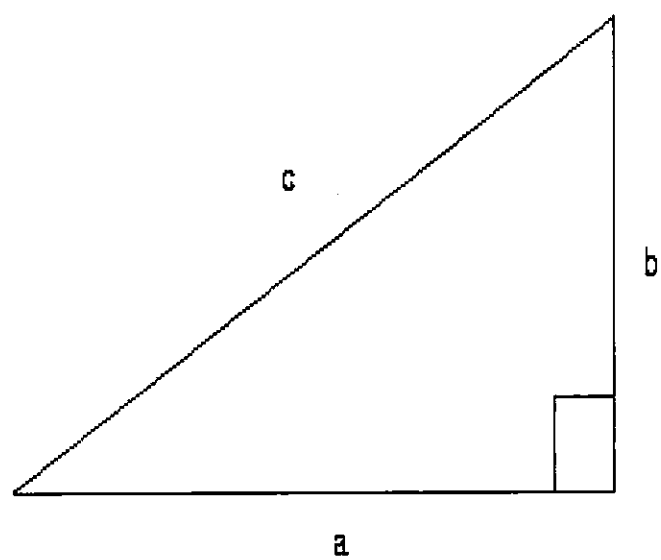

Figure 1. The Pythagorean theorem, $a^2 + b^2 = c^2$.

These scales are not logarithmic scales. Since the scales have no obvious relation to the C scale, we will instead relate the P, Q, and Q' scales to the L scale. The L and P scales both range from 0 to 1. If we set the cursor to l on the L scale, and read the corresponding value p from the P scale, we find that $p = \sqrt{l}$ or $l = p^2$. Note that the P scale can be used with the L scale to compute squares and square roots. The Q scale on the slide is identical to the P scale on the body of the rule. The Q' scale on the slide is an extension to the Q scale. If we set the cursor to l on the L scale, then the corresponding value on the Q' scale is given by $q' = \sqrt{1 + l}$. The Q' scale runs from 10.0 to 14.14. The relationships between these scales are summarized in Table 2.

Suppose that we know a and b and wish to determine c. Set the left index of the Q scale under a on the P

Hemmi 152	L, K, A / B, CI, C / D, T θ, Rθ, P / Q, Q', C / LL3, LL2, LL1
Hemmi 153	L, K, A / B, CI, C / D, T, Gθ θ, Rθ, P / Q, Q', C / LL3, LL2, LL1
Hemmi 154	Sx, Tx, Tθ / Th, Sh, Sh, C / D, L, X DF, P', P / Q, CF, CI, Sθ / A, D, K
Lafayette Vectorlog	Tr1, Tr2, P', P / Q, ST, Sr, Sθ, C / D, LL03, LL02, LL01 Sh1, Sh2, DF, A / B, CF, Th, CI, C / D, LL3 LL2, db
Pickett 110–ES	EI (1&2), B, CI, C // D, A, K, LL0, LL1, LL3, LL4 D(4), D, V2 // V1, C(4), θ, tan, sin, tan, sin, tan, tan & sin, L
Relay 157	Sr, Sθ, P', P / Q, CF, CI, C / D, DF, LL1, LL2, LL3 Sh2, Sh1, A / B, K, Th, C / D, Tr1, Tr2, dB
Relay 158	Sh2, Sh1, Th, A / BI, S, T, CI, C / D, LL3, LL2, LL1 X2, X1, P2, P1 / Q, Y, L / x, I, I3, q1, q2, y

Table 1. Slide Rules using the Hemmi System.

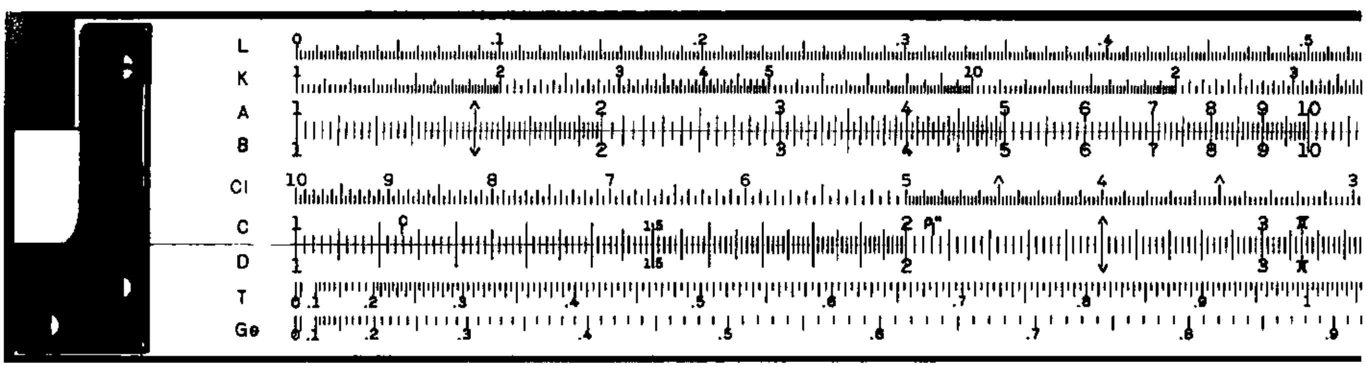

Figure 2a. Left end of the face of a Hemmi 153 slide rule.

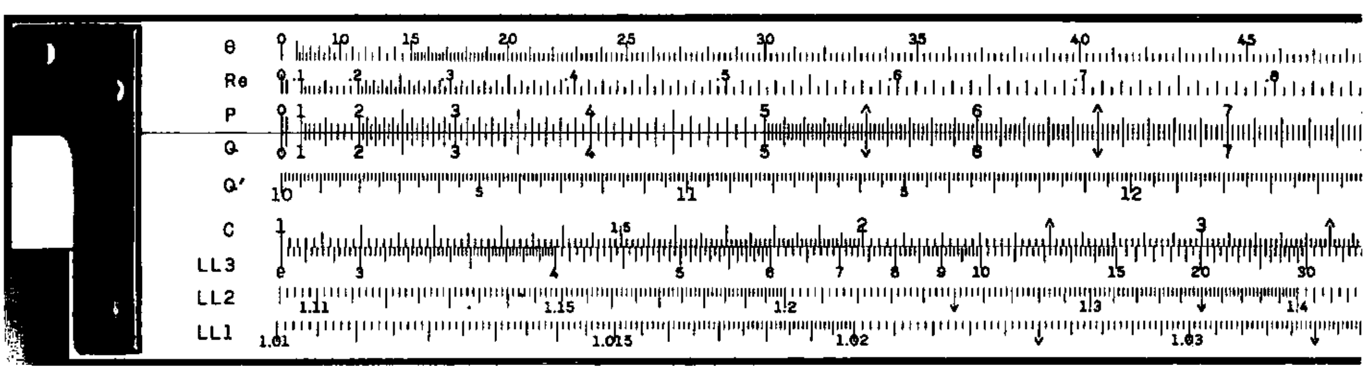

Figure 2b. Left end of the back of a Hemmi 153 slide rule.

scale. This corresponds to a^2 on the L scale. Then move the cursor over to b on the Q scale. This corresponds to adding b^2 to a^2 on the L scale. Thus we can read c under the cursor on the P scale. Although the P and Q scales run from 0 to 10, it is possible to use these scales for other ranges of values as long as the lengths of the sides are all of the same order of magnitude.

For example, consider the problem of finding the hypotenuse of a right triangle with sides of length 3 and 4. Set the left index of the Q scale under 3 on the P scale. Move the cursor to 4 on the Q scale. Then read off the length of the hypotenuse, 5, on the P scale.

Things become slightly more complicated when $a^2 + b^2$ is off scale. One option is simply to scale a and b so that $a^2 + b^2$ will remain on scale. A better approach is to make use of the Q' scale. To use the Q' scale, set a on the Q scale under the right index of the P scale. Then move the cursor to b on the P scale, and read c under the cursor on the Q' scale. For example, consider the problem of finding the hypotenuse of a right triangle with sides 9 and 7. Set 9 on the Q scale under the right index of the P scale. Move the cursor to 7 on the P scale and read the length of the hypotenuse, 11.4, under the cursor on the Q' scale.

The above procedures can easily be reversed to obtain a from b and c. For example, suppose that $c = 9.5$ and $b = 5.0$. Set the cursor to 9.5 on the P scale. Set 5.0 on the Q scale under the cursor. Read $a = 0.81$ on the P scale over the left index of the Q scale. This calculation can also be accomplished by setting the left index of Q under b on the P scale and then reading a on the Q scale under c on the P scale.

One particularly important case is the evaluation of $\sqrt{1-x^2}$. To perform this calculation, set the left index of the Q scale under x on the P scale and read the answer on the Q scale under the right index of the P scale. Because the P and Q scales are not related to the C scale, this is not equivalent to using the P scale commonly found on European slide rules. In general, the European style P scale produces more accurate answers for very small values of x.

Trigonometric Functions

The Hemmi system works with angles in both decimal degrees and radians. The Rθ scale is used for angles in radians while the θ scale is used for angles in degrees.

Conversions between radians and degrees can be done by simply setting the cursor to the angle on the Rθ or θ scale and then reading off the angle from the other scale. For angles of less than about 7.5°, a more accurate conversion from degrees to radians can be done by multiplying the angle by ten, converting to radians, and then dividing the result by 10. For example, to convert 6° to radians, set the cursor to 60° on the θ scale. Read the angle of 0.1047 radians off the Rθ scale.

If we set the cursor to an angle θ on the Rθ or θ scale, and read the corresponding value l on the L scale, then $l = \sin^2\theta$ and $\theta = \sin^{-1}\sqrt{l}$. Thus we can use the R$\theta$ and θ scales along with the L scale to compute $\sin^2\theta$. Since $\sin^2\theta + \cos^2\theta = 1$, a simple subtraction gives $\cos^2\theta$.

The θ and Rθ scales can be used with the P scale to evaluate sines of angles. If we set the cursor at an angle θ, then the corresponding value of l on the L scale is given by $l = \sin^2\theta$. If we look at the corresponding value of p on the P scale, we find that $p = \sqrt{l}$, or $p = \sin\theta$. Since $\cos\theta = \sqrt{1-\sin^2\theta}$, we can compute the cosine of θ by first computing $\sin\theta$ and then using the P and Q scales to compute $\cos\theta = \sqrt{1-\sin^2\theta}$.

For example, to find the sine of 40°, set the cursor at 40° on the θ scale. Read the sine of the angle, 0.643, under the cursor on the P scale. Next, put the index of the Q scale under the cursor. Read the cosine of the angle, 0.766, on the Q scale under the right index of the P scale.

For angles less than about 7.5°, a more accurate approach is to convert the angle to radians using the approach described previously and then use the approximation $\sin\theta \approx \theta$ to approximate $\sin\theta$. For example, to compute the sine of 6.5°, set the cursor to 65° on the θ scale. Then read the angle in radians, 0.1135, from the Rθ scale. This is a very good approximation to $\sin 6.5°$, which is actually 0.1132.

If we set the cursor to an angle θ on the Rθ or θ scales, then the corresponding value of l on the L scale is given by $l = \sin^2\theta$. The T scale is calibrated so that l on the L scale corresponds to $t = \tan\sin^{-1}\sqrt{l}$ on the T scale. Thus $t = \tan\theta$. For example, to evaluate $\tan 40°$, set the cursor to 40° on the θ scale. Then read the tangent, 0.839, under the cursor on the T scale.

The Rθ and θ scales are referenced to the L scale while the conventional S scale is referenced to the logarithmic C scale. As a result there are some interesting differences in the resolution of the scales. On the Hemmi 153's θ scale, there are marks at each half degree up to 85 degrees. On the S scale of a post versalog slide rule, there are marks at each degree up to 80 degrees. On the other hand, Hemmi 153's T scale must cover a range of 90 degrees, while the conventional T scale only covers the range from 5.6 to 45 degrees. Thus the conventional T scale makes it easier to obtain accurate tangents of angles.

The P, P′, Q, Sr, and Sθ scales of the Lafayette F–686 and 99-71029 Vectorlog slide rule are similar to the scales of the Hemmi 153. The Lafayette slide rules also have a conventional ST scale and a two-part scale for tangents of angles in radians.

Hyperbolic Functions

The system of scales for hyperbolic functions is based on the Gudermannian function gd x. The Gudermannian is defined by

$$\text{gd}\,x = 2\tan^{-1}\left(\tanh\frac{x}{2}\right).$$

It can also be shown that an equivalent definition is

$$\text{gd}\,x = \sin^{-1}(\tanh x).$$

A plot of the function is shown in Figure 3. Further information on the Gudermannian function can be found in [3].

Although the Gudermannian looks obscure, the function has special properties that connect trigonometric functions and hyperbolic functions. The identities are:

$$\begin{aligned}\sinh x &= \tan \mathrm{gd}\, x\\ \cosh x &= \sec \mathrm{gd}\, x\\ \tanh x &= \sin \mathrm{gd}\, x\end{aligned}$$

Note that in these identities, all trigonometric functions are in radians. Using these identities, it is possible to turn any calculation involving hyperbolic functions into a calculation involving the gd x function and trigonometric functions.

Scale	From L to Scale	From Scale to L	Comments
P	$p = \sqrt{l}$	$l = p^2$	
Q	$q = \sqrt{l}$	$l = q^2$	
Q'	$q' = \sqrt{1+l}$	$l = q'^2 - 1$	
θ	$\theta = \sin^{-1}\sqrt{l}$	$l = \sin^2\theta$	θ in degrees
Rθ	$\theta = \sin^{-1}\sqrt{l}$	$l = \sin^2\theta$	θ in radians
T	$t = \tan\sin^{-1}\sqrt{l}$	$l = (\sin\tan^{-1}t)^2$	$t = \tan\theta$
Gθ	$g = \tanh^{-1}\sqrt{l}$	$l = (\tanh l)^2$	$\theta = \mathrm{gd}\, g$

Table 2. Relationships between scales.

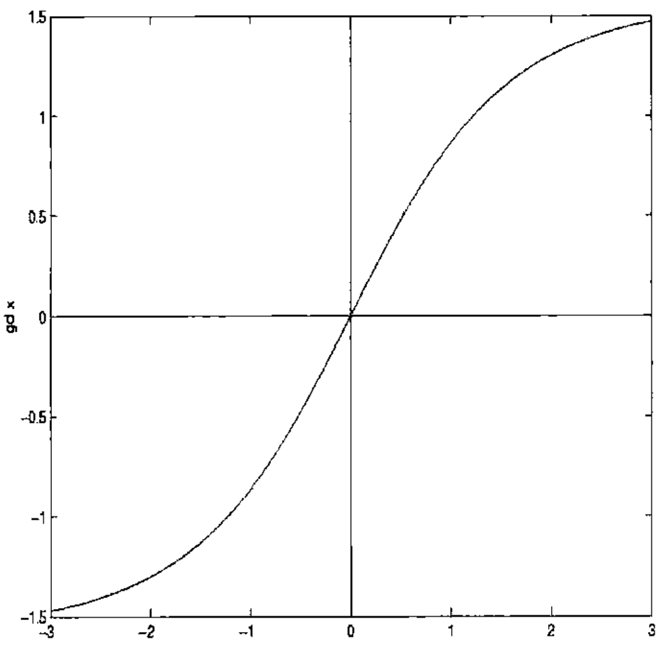

Figure 3. The Gudermannian function gd x.

If we set the cursor to l on the L scale, the corresponding point g on the Gθ scale is given by $g = \tanh^{-1}\sqrt{l}$. This relationship can also be written as $l = (\tanh g)^2$. The point θ on the Rθ scale corresponding to l is given by $\theta = \sin^{-1}\sqrt{x}$. Thus

$$\theta = \sin^{-1}(\tanh g)$$

and

$$\theta = \mathrm{gd}\, g.$$

The Gθ scale ranges from 0 to 6, but has poor resolution for arguments less than 0.1 or greater than 3. This is comparable to the range and resolution of convention Sh1 and Sh2 scales. Since gd $-x = -$gd x, we can also handle negative arguments.

Now we can use the identities relating the Gudermannian to trig functions to evaluate hyperbolic functions. To find $\sinh z$, set the cursor to z on the Gθ scale and read $\sinh z = \tan \mathrm{gd}\ z$ on the T scale. To find $\tanh z$, set the cursor to z on the Gθ scale and read $\tanh z = \sin \mathrm{gd}\ x$ on the P scale. To find $\cosh z$, set the cursor to z on the Gθ scale, and use the P and Q scales to find $\cos \mathrm{gd}\ x$. Transfer this number to the C scale, and then read its reciprocal, $\sec \mathrm{gd}\ x = \cosh z$, from the CI scale.

For example, to find sinh 2 and tanh 2, set the cursor to 2 on the Gθ scale. Read $\sinh 2 = 3.6$ from the T scale. Then read $\tanh 2 = 0.96$ from the P scale.

For arguments outside of the range of the Gθ scale, a number of approximations are available. Table 3 gives a list of useful approximations.

$\sinh x \approx x$	$x < 0.1$
$\sinh x \approx 0.5e^x$	$x > 3.0$
$\tanh x \approx x$	$x < 0.1$
$\tanh x \approx 1$	$x > 3.0$
$\cosh x \approx 1$	$x < 0.1$
$\cosh x \approx 0.5e^x$	$x > 3.0$

Table 3. Approximations for hyperbolic functions.

The Lafayette Vectorlog slide rules have conventional Sh1, Sh2, and TH scales in place of the Gθ scale of the Hemmi 153. These scales have the same useful range (0.1 to 3.0) as the Gθ scale, but the Sh1 and Sh2 scales allow for much greater accuracy in computing the hyperbolic sines of small arguments.

Conclusions

We have described the Sun Hemmi system of scales for trigonometric, and hyperbolic functions. The system is extremely flexible, and particularly good for solving right triangles. However, it is also more complicated than the more conventional system of S, T, ST, Sh1, Sh2, and Th scales. Although the system was also partially adopted by Hughes Owens, Relay, Ricoh, and Lafayette, it was not adopted by manufacturers in the US or Europe. Furthermore, Hemmi used conventional scales for trigonometric and hyperbolic functions in many other models of its slide rules.

References

[1] Bill Lise. Personal Communication. October 3, 2000.

[2] Hisashi Okura. Hyperbolic scale rule. US Patent Number 2,079,464, 1937.

[3] Daniel Zwillinger, editor. *CRC Standard Mathematical Tables and Formulae, 30th ed.* CRC Press, Boca Raton, 1996.